Comments

Further Comments on "Perturbations in the Ionosphere Caused by a Moving Body"

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In a recent note Browand[1] noted some discrepancies in a paper by Gurevich,[2] and, in addition, formulated the problem for the determination of the density distribution behind a satellite of circular cross section in free molecule flow. Browand[1] also gave some approximate numerical results for the density distribution, noted the good agreement with Gurevich's numerical results, but was unable to form any comparison between the two.

The purpose of the present note is to show the relation between an exact formulation of the problem as indicated by Browand and as given in some detail below, and the approximate results presented by Gurevich.

Before making a comparison of results the authors would like to present a solution in cylindrical coordinates. The circular disk of radius R_0 is located with its center at the origin of a cylindrical coordinate system (r,θ,z) with z positive downstream from the disk. If one starts from the steady-state, collisionless Boltzmann equation (ignoring external forces) in Cartesian coordinates (x,y,z,c_x,c_y,c_z) and makes a transformation of independent variables to the cylindrical coordinates $(r,\theta,z,c_r,c_\theta,c_z)$, then one has

$$c_r\frac{\partial f}{\partial r} + \frac{c_\theta}{r}\frac{\partial f}{\partial \theta} + \frac{c_\theta{}^2}{r}\frac{\partial f}{\partial c_r} - \frac{c_r c_\theta}{r}\frac{\partial f}{\partial c_\theta} + c_z\frac{\partial f}{\partial z} = 0 \qquad (1)$$

Since the problem has axial symmetry, f is independent of θ and the second term vanishes giving the same equation as in Ref. 1. Now transforming from the variables (r,z,c_r,c_θ,c_z) to the independent variables (r,z,c,φ,c_z), Eq. (1) becomes

$$c\cos\varphi\frac{\partial f}{\partial r} - \frac{c}{r}\sin\varphi\frac{\partial f}{\partial \varphi} + c_z\frac{\partial f}{\partial z} = 0 \qquad (2)$$

where $c^2 = c_r{}^2 + c_\theta{}^2$ and φ is the angle between c and c_r. The general solution of Eq. (2) is obtained from the characteristics in the form:

$$f = f[r^2 - (2/c_z)crz\cos\varphi + c^2z^2/c_z{}^2;\ c,c_z] \qquad (3)$$

It is interesting to note that if the Boltzmann equation were written symbolically as

$$\mathbf{c}\cdot(\partial f/\partial \mathbf{r}) + c_z(\partial f/\partial z) = 0 \qquad (4)$$

then the solution could be written as $f = f(\mathbf{r} - \mathbf{c}z/c_z; c,c_z)$, which is equivalent to Eq. (3).

The boundary condition is specified in the plane of the disk ($z = 0$), where the gas is assumed to have a Maxwellian distribution with mean velocity c_0. That is, $f = f_m$ for $|\mathbf{r}| > R_0$, and $f = 0$ for $|\mathbf{r}| < R_0$, where

$$f_m = n_0(m/2\pi kT)^{3/2}\exp\{-m/2kT[c^2 + (c_z - c_0)^2]\}$$

The solution of Eq. (4) or (2) subject to this boundary condition in the region $z > 0$ is

$$f = \tfrac{1}{2}f_m[1 + \operatorname{sign}(|\mathbf{r} - \mathbf{c}z/c_z| - R_0)] \qquad (5)$$

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where sign $A = 1$ for $A > 0$, sign $A = -1$ for $A < 0$. The density $n(r,z)$ is obtained by integrating Eq. (5) over all velocity space, except as restricted by the arguments of the sign function. The integration will be carried out using cylindrical coordinates in velocity space c,φ,c_z. Before writing out the density explicitly it will be worthwhile to determine the limits on c_z as required by the sign function. This is accomplished by setting $|\mathbf{r} - \mathbf{c}z/c_z| = R_0$. Squaring and introducing nondimensional quantities $\rho = r/R_0$, $Z = z/R_0$, the following result is obtained:

$$\rho^2 - (2/c_z)Z\rho c\cos\varphi + c^2Z^2/c_z{}^2 = 1 \qquad (6)$$

Solving this expression for c_z gives the values which are to be used as limits in evaluating the density integral. The two roots of Eq. (6) are

$$c_z = -Zc(1 - \rho^2)^{-1}[\rho\cos\varphi \pm (1 - \rho^2\sin^2\varphi)^{1/2}] \qquad (7)$$

For the case $\rho < 1$, it is clear that if $|\mathbf{r} - \mathbf{c}z/c_z| \geqslant R_0$, then c_z cannot be infinite; consequently, the limits of integration for c_z will be between the two values given in Eq. (7) for all values of φ.

For $\rho > 1$ the situation is a little more complicated because as Eq. (7) indicates $\sin^2\varphi > 1/\rho^2$ in order that the limits remain real. After some investigation one can show that the ranges for c_z for which $f \neq 0$ are $-\infty < c_z < \infty$ for $\sin^{-1}(1/\rho) < \varphi < 2\pi - \sin^{-1}(1/\rho)$, and $-\infty < c_z < G$, $H < c_z < \infty$ for $-\sin^{-1}(1/\rho) < \varphi < \sin^{-1}(1/\rho)$, where

$$G = [Zc/(\rho^2 - 1)][\rho\cos\varphi - (1 - \rho^2\sin^2\varphi)^{1/2}]$$

and

$$H = [Zc/(\rho^2 - 1)][\rho\cos\varphi + (1 - \rho^2\sin^2\varphi)^{1/2}]$$

The density for $\rho < 1$ can now be expressed as follows,

$$n/n_0 = \pi^{-3/2}\int_E^F\int_0^{2\pi}\int_0^\infty \exp\{-[V^2 + (V_z - V_0)^2]\}VdVd\varphi dV_z \qquad (8)$$

where $V = c/(2kT/m)^{1/2}$, $V_z = c_z/(2kT/m)^{1/2}$, with E and F denoting the expressions from Eq. (7) corresponding to the positive and negative signs, respectively. For $\rho > 1$, the density is

$$\frac{n}{n_0} = \pi^{-3/2}\left\{\int_{-\infty}^{\infty}\int_B^{2\pi-B}\int_0^\infty I + \int_{-\infty}^{G}\int_{-B}^{B}\int_0^\infty I + \int_H^\infty\int_{-B}^{B}\int_0^\infty I\right\} \qquad (9)$$

where $B = \sin^{-1}(1/\rho)$ and the integrand I is the same as that of Eq. (8).

The evaluation of Eq. (8) can be carried out by first integrating with respect to V_z and then integrating by parts with respect to V, thus yielding

$$n/n_0 = (4\pi)^{-1}\int_0^{2\pi}\{a(1 + a^2)^{-1/2}\exp[-V_0{}^2/(1 + a^2)] \times \{1 + \operatorname{erf}[aV_0/(1 + a^2)^{1/2}]\} - b(1 + b^2)^{-1/2}\exp[-V_0{}^2/(1 + b^2)] \times \{1 + \operatorname{erf}[bV_0/(1 + b^2)^{1/2}]\}\}d\varphi \qquad (10)$$

where

$$a = Z(1 - \rho^2)^{-1}[(1 - \rho^2\sin^2\varphi)^{1/2} - \rho\cos\varphi]$$

and

$$b = -Z(1 - \rho^2)^{-1}[(1 - \rho^2\sin^2\varphi)^{1/2} + \rho\cos\varphi]$$

The first integral in Eq. (9) can be easily evaluated, while the procedure just described can be applied to the last two integrals giving

$$\frac{n}{n_0} = 1 + (4\pi)^{-1}\int_{-B}^{B} J \qquad (11)$$

where the integrand J is the same as the integrand in Eq. (10).

Equations (10) and (11) have been numerically integrated and the results are plotted in Fig. 1. Also shown for comparison is the sketched curve given by Browand for $n/n_0 = 0.95$. Along the centerline $r = 0$ an expression for the density in terms of elementary functions can be given, since then $a = Z$ and $b = -Z$, and so Eq. (10) reduces to

$$n/n_0 = Z(1 + Z^2)^{-1/2} \exp[-V_0^2/(1 + Z^2)] \quad (12)$$

As just mentioned, the expression for the density given by Gurevich[2] is approximate. Unfortunately, Ref. 2 contains no statement of the approximation. However, in another paper Gurevich[3] considers the same problem and assumes that the molecules possess no thermal component of velocity in the z direction.‡ This means that in Eq. (5) the c_z that appears in the argument of the sign function is replaced by c_0, but the exponential is unchanged. This is necessary so that the freestream density at $z = 0$ can be denoted by n_0. The c_z integration can then be done immediately. Since the argument of the sign function is just the characteristic of the differential equation, it represents the trajectories of the molecules from the plane $z = 0$ into the region of interest ($z > 0$) and corresponds to the assumption of constant velocity c_0 in the z direction. One can then proceed to carry out the remaining two integrals by a method analogous to that just used, and by making use of a transformation similar to that given by Kornowski[4] in the study of the nonsteady expansion of a gas into an initially evacuated, infinitely long cylinder. The integral for the density given by Gurevich [Eq. (9), Ref. 3] is correct based on the assumption just mentioned and has been tabulated by Brinkley[5] and by Masters.[6]

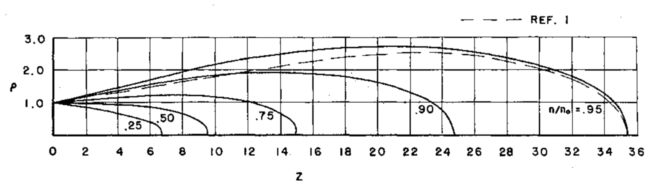

Fig. 1 Constant density lines for $V_0 = 8$.

Gurevich's expression for centerline density is

$$n/n_0 = \exp[-(V_0/Z)^2] \quad (13)$$

and comparing this with Eq. (12), it is clear that the two expressions are nearly the same, except for small values of Z. This explains the relatively good agreement between the exact calculations shown in Fig. 1 and Gurevich's approximate results. A complete discussion of this and related problems will soon be available as a Purdue University report.

References

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‡ Ref. 3, p. 329.

Temperature Distribution in a Case-Bonded Cylindrical Rocket Assembly

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Introduction

When a case bonded propellant grain is subjected to changes in thermal environment, bond stresses develop as a result of a differential thermal expansion between the casing and the propellant grain. The complexity of the bond stress equation depends upon the prescribed temperature condition. In the case-bonded propellant grain problem, the transient temperature expression is a complicated function of space and time so that the corresponding bond stress equation is extremely cumbersome. To simplify the bond stress expression for use in engineering analyses, studies were recently presented[1,2] in which the temperature of the propellant grain was chosen to be either equal to the original equilibrium temperature or equal to the temperature of the new environment. These simple temperature conditions provide simple bond stress equations which correspond to the worst stress conditions in an elastic material.[3]

The forementioned simplifying conditions are no longer satisfactory for a viscoelastic analysis of the problem, because the resulting stresses are both time and space dependent. It is suggested that the difficulties now be overcome by assuming that the temperature of the propellant grain is independent of the radial coordinate but varies only with the time coordinate. Justification for the proposed temperature variation will be discussed in the present paper. The associated viscoelastic solution has been presented elsewhere.[4]

Temperature Distribution

The cross section of a typical solid propellant rocket assembly is shown in Fig. 1. It may have either a solid propellant grain of external radius b or a hollow propellant grain of internal radius a and external radius b, surrounded by either a metallic or plastic casing that is physically thin in comparison to its external radius c. From a heat-transfer point of view, the outer surface of the assembly has a convective heat-transfer coefficient h_0, and the bond surface between the propellant grain and the casing has a thermal resistance h_i.

A general equation for the temperature distribution within the propellant grain is, practically, difficult to write under arbitrary boundary conditions because of the complexity of the algebra involved. If, however, the range of ambient temperature variation and heat-transfer parameters is restricted, a solution may be found which lends itself to further use in viscoelastic analysis.

From considerations of elementary heat transfer, one can write an expression for the temperature-time distribution in a cylindrical grain which is initially at a uniform temperature T°, but whose temperature is altered when the ambient temperature changes to zero at time $t = 0$, namely,[5]

$$T(r,t) = 2T^\circ \sum_{n=1}^{\infty} \left[\frac{J_0(\lambda_n r)J_1(\lambda_n b)}{J_0^2(\lambda_n b) + J_1^2(\lambda_n b)}\right] \frac{e^{-\alpha t \lambda_n^2}}{\lambda_n b} \quad (1)$$

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